CS 259C/Math 250: Elliptic Curves in Cryptography
Homework 3 Solutions

1. The basic idea of the new signature scheme is that (e, s) can be computed from (R, s) and vice versa, given M. Given M and $\sigma = (R, s)$, we can compute $\sigma' = (e, s)$ where

$$e = H(M||R)$$

Given M and $\sigma' = (e, s)$, we can compute $\sigma = (R, s)$ where

$$R = [s]P - [e]Q = [s - ae]P = [k]P = R$$

Hence, any party can convert a signature from Sign to one from Sign' and vice versa.

(a) According to the note above, $\mathsf{Verify}'(pk, M, \sigma') = \mathsf{Verify}(pk, M, \sigma)$ where $\sigma = ([s]P - [e]Q, s)$. Specifically, $\mathsf{Verify}'(pk, M, \sigma)$ works as follows:

1. Compute $R = [s]P - [e]Q$
2. Compute $e' = H(M||R)$
3. Accept if $R + [e']Q = [s]P$. Note that this condition is equivalent to $e = e'$

(b) Suppose we have an adversary A for Sign' with advantage ϵ. We construct B, and adversary for Sign, as follows:

1. On input pk, simulate A on pk.
2. When A asks for a signature on M, B asks its challenger for a signature on M. When the challenger responds with $\sigma = (R, s)$, compute $e = H(M||R)$ and send $\sigma' = (e, s)$ to A.
3. When A returns a forgery candidate (M, σ') where $\sigma' = (e, s)$, B returns (M, σ) where $\sigma = (R, s)$ and $R = [s]P - [e]Q$.

To show that the signatures seen by A are from the same distribution as signatures from Verify', note that the signature queries are answered as follows:

1. B's challenger chooses a random k and computes $R = [k]P$
2. B's challenger computes $e = H(M||R)$
3. B's challenger sets $s = k + ae$, and sends (R, s) to B.
4. B computes $e' = H(M||R) = e$
5. B returns $\sigma' = (e', s) = (e, s)$ to A.

Hence, this computation is equivalent to the computation of σ' from Verify'. This means that the view of A as a subroutine of B is identical to that as an adversary for the modified scheme. Thus, A outputs a valid forgery as a subroutine of B with probability ϵ.

If A outputs a valid forgery $(M, \sigma' = (e, s))$, it means that A never asked for a signature on M and that $\mathsf{Verify}(pk, M, \sigma')$ accepts. But this means that B also never asked for a signature on M and that $Ver(pk, M, \sigma = ([s]P - [e]Q, s)) = Ver'(pk, M, \sigma' = (e, s))$ accepts. Hence, (M, σ) is a valid forgery for Sign. Thus, B outputs a valid forgery with probability ϵ, so its advantage is ϵ.

(c) This scheme has the advantage that signatures are two integers mod r (which takes $2 \log r$ bits to represent) as opposed to a point on the curve and a integer mod r (which takes $2 \log q + \log r$ bits if we naively encode the point using its x and y coordinates) in the original scheme. Even if we compress the representation of the point to $2 \log q + 1$ bits, the modified scheme will still have shorter signatures when r is smaller than q.

2. (a) If an adversary could compute a k such that $R = [k]G$ with x coordinate 0, then a valid signature on a document m would be

$$(R, k^{-1}(m + ax) \bmod r) = (R, k^{-1}m \bmod r)$$

This is easily computable for any m.

(b) If $s = 0$, then $k^{-1}(m + ax) \bmod r = 0$. This means $a = m/x \bmod r$.

3.

Time to solve DLP	Size of p for $E(\mathbb{F}_p)$	Size of p for $\mathbb{F}_p^\times$
2^{56}	2^{112}	2^{383}
2^{80}	2^{160}	2^{853}
2^{112}	2^{224}	2^{1859}
2^{128}	2^{256}	2^{2547}
2^{192}	2^{384}	2^{6732}
2^{256}	2^{512}	2^{13599}

4. (a) We know from the previous homework that if $E(\mathbb{F}_p)$ is supersingular for a prime p, then either $E(\mathbb{F}_p)$ is cyclic or isomorphic to

$$\mathbb{Z}_2 \times \mathbb{Z}_{\frac{p+1}{2}}$$

In the latter case, the entire 2-torsion is contained in $E(\mathbb{F}_p)$. Recall from homework 1 that elements of order 2 are points with y coordinate 0. The x coordinates are thus solutions to

$$x^3 + x = 0$$

One solution is $x = 0$, and the other two are solutions to $x^2 + 1 = 0$. But -1 is not a square mod p (since p is 3 mod 4). Therefore, the only point of order 2 in $E(\mathbb{F}_p)$ is $(0, 0)$, meaning the 2-torsion is *not* contained in $E(\mathbb{F}_p)$. Thus $E(\mathbb{F}_p)$ is cyclic.

(b)-(f) I wrote a routine that solves the discrete log problem mod a given integer, assuming that integer divides the order of P.

```
def DLSolve(P,Q,n):
    '''Solve for a mod n where Q=aP, assuming n divides the order of P'''
    k=P.order()
    r = floor(k/n)
    PP=r*P
    QQ=r*Q
    return PP.discrete_log(QQ)

b = DLSolve(P,Q,2); print(b)
c = DLSolve(P,Q,4); print(c)
d = DLSolve(P,Q,3); print(d)
e = DLSolve(P,Q,41); print(e)

a = CRT([c,d,e,s],[4,3,41,t]); print(a)
Q == a*P
```

```
1
1
2
39
7771731116344866325082308703881561488257139696 41
True
```

For part (c) we could also have solved for a mod 4 using part (b). Part (b) tells us that a is odd, so we can write $a = 2a' + 1$. Then defining $Q' = Q - P$, we have $Q' = a'(2P)$. We can then solve for a' mod 2, and find that it is equal to 0. This tells us that $a = 1$ mod 4.

5. (a) Let m be some integer such that $m^2 \geq w + 1$. Compute and save $[b+i]P$ for all $i \in [0, m-1]$. Now, for each $j \in [0, m-1]$, compute $Q - [mj]P$, and check if it matches one of the stored values. If we have a match, we have

$$[b+i]P = Q - [mj]P$$

Therefore $[b + mj + i]P = Q$, so $a = b + mj + i$.

This scheme uses $\log b$ group operations to compute $[b]P$, and then $m \approx \sqrt{w}$ group operations to compute $[b+i]P$ for each i. Then we need to compute $[m]P$ using $\log m$ group operations, and we compute $Q - [mj]P$ for each j using another $m \approx \sqrt{w}$ operations. $\log m \approx \frac{1}{2} \log w$ which is much smaller than $\sqrt{w}$. Also, $\log b$ is most likely much smaller than $\sqrt{w}$. Therefore, the total number of group operations is about $2\sqrt{w}$.

This scheme works because we can write $a = b + a_0 + ma_1$ for some $0 \leq a_0, a_1 < m$. When $i = a_0$ and $j = a_1$,

$$Q - [mj]P = [a]P - [ma_1]P = [a_0]P = [i]P$$

So we have found a match.

Alternatively, we can just compute $Q' = Q - [b]P$, and solve $Q' = [a']P$ with a' in the interval $[0, w]$ using the standard baby step-giant step algorithm with an upper bound w.

(b) First, compute $Q' = Q - [t]P$, which equals $[a-t]P$. Letting $\tilde{a} = a - t$, we are now solving the problem of computing $\tilde{a}$ where $Q' = [\tilde{a}]P$ and $\tilde{a} \equiv 0$ mod m. That is, $\tilde{a} = m\ell$ for some ℓ

Next, compute $P' = [m]P$. We now have $Q' = [a']P'$ where $a' = \frac{\tilde{a}}{m} = \ell$. We know that

$$a' = \frac{\tilde{a}}{m} = \frac{a-t}{m} < \frac{r}{m}$$

Therefore, we have reduced this problem to finding the discrete log on an interval of length approximately r/m.

6. (a)

```
def floyd_rho(P,Q):
    '''Compute discrete log using Floyd cycle finding.'''


    # Initialize as above.

    n = P.order()

    walk = walk_setup(P,Q) # set up the walk function

    u0 = randint(1, P.order())
    Xi = (u0*P, u0, 0)
    X2i = Xi

    nWalkCalls = 0

    # Repeat until P_i = P_{2i}

    while True:
        # Compute P_{i} and P_{2i}

        Xi = walk(Xi)
        X2i = walk(walk(X2i))
        nWalkCalls += 3

        (Pi, ui, vi) = Xi
        (P2i, u2i, v2i) = X2i

        if Pi == P2i:
            if (v2i-vi) % n != 0:
                a = ((ui-u2i)/(v2i-vi)) % n
                return a,nWalkCalls
            else:
                u0 = randint(1, P.order())
                Xi = (u0*P, u0, 0)
                X2i = Xi
```

(b)-(c)

```
def distpt_rho(P,Q,d):
    '''Compute discrete log of Q to the base P using distinguished points.
       1/d = probability of hitting a distinguished point'''

    # Initialize as above.

    n = P.order()

    walk = walk_setup(P,Q) # set up the walk function

    u0 = randint(1, P.order())
    Xi = (u0*P, u0, 0)
    nWalkCalls = 0
    iterations = 0
    D = {}

    while(True):
        # Iterate the random walk

        Xi = walk(Xi)
        nWalkCalls += 1
        iterations += 1
        (Pi, ui, vi) = Xi

        x = Pi[0]

        # Test for distinguished points
        if ZZ(x)%d == 0:   # here x is the x-coordinate of P_i

            if Pi in D:
                (u, v) = D[Pi]
                if ((v-vi) % n) != 0:
                    a = ((ui-u)/(v-vi)) % n
                    return a,len(D),nWalkCalls

            else:
                D[Pi] = (ui,vi)
                u0 = randint(1, P.order())
                Xi = (u0*P, u0, 0)
                iterations = 0
        elif iterations > 100 * d:
                u0 = randint(1, P.order())
                Xi = (u0*P, u0, 0)
                iterations = 0
```

(d)

```
print(mean([collision_search(P,Q)[1].N() for i in range(1000)])/sqrt(P.order()).N())
print(mean([floyd_rho(P,Q)[1].N() for i in range(1000)])/sqrt(P.order()).N())
print(mean([distpt_rho(P,Q,32)[2].N() for i in range(1000)])/sqrt(P.order()).N())
```

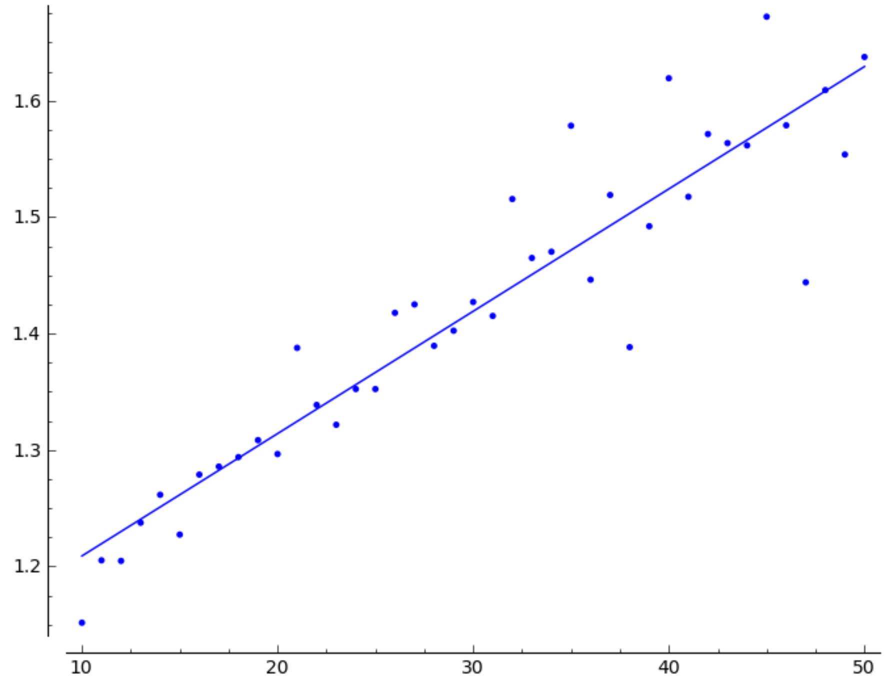
```
1.33860182429175
3.23329579472458
1.44856888334897
```

(e)

```
var('b, c, x')
drange = range(10,51,1)
data = [(d,mean([distpt_rho(P,Q,d)[2].N() for i in range(500)])/sqrt(P.order()).N())
for d in drange]
model(x) =b*x+c
find_fit(data,model)
```

```
[b == 0.010504814963271714, c == 1.1039768365480975]
```

```
show(plot( 0.010504814963271714*x+1.1039768365480975,(x,10,50))+list_plot(data))
```



Notice that this plot seems roughly linear in d.

7. (a)

$$\begin{aligned}
\hat{f}(-R) &= \hat{f}(x,-y) = \begin{cases} f(x,-y) & \text{if } -y < y \pmod p \\ f(x,y) & \text{if } -y > y \pmod p \end{cases} \\
&= \begin{cases} f(x,y) & \text{if } y > -y \pmod p \\ f(x,-y) & \text{if } y < -y \pmod p \end{cases} \\
&= \hat{f}(x,y)
\end{aligned}$$

(b) If $\hat{P}_i$ and $\hat{P}_j$ have the same x-coordinate, then $\hat{P}_i = \pm\hat{P}_j$, and we can tell weather it is a plus or minus (by comparing y-coordinates). Thus, we have

$$u_iP + v_iQ = \pm(u_jP + v_jQ)$$

This can be rearranged as (assuming $v_i \neq \pm v_j$):

$$Q = -\frac{u_i \mp u_j}{v_i \mp v_j}P$$

Hence, we have computed the discrete log.

(c) Before, we were in a space of N objects, the points on the elliptic curve. Now we are in a space of about $N' = N/2$ objects, the x-coordinates of those points. Thus, the average number of iterations is about $c\sqrt{N'} = c\sqrt{N/2}$

(d) Recall that $f(P) = P + M_{x \bmod s}$. Further, $y_{i+1} > -y_{i+1}$, so $\hat{f}(P_{i+1}) = f(-P_{i+1})$. Thus

$$\begin{aligned} \hat{P}_{i+2} &= \hat{f}(\hat{P}_{i+1}) = f(-\hat{P}_{i+1}) = -\hat{P}_{i+1} + M_{x_{i+1} \bmod s} \\ &= -\hat{f}(\hat{P}_i) + M_{x_{i+1} \bmod s} = -f(\pm\hat{P}_i) + M_{x_{i+1} \bmod s} \\ &= \mp\hat{P}_i - M_{x_i \bmod s} + M_{x_{i+1} \bmod s} = \mp\hat{P}_i \end{aligned}$$

Therefore, $\hat{P}_i$ and $\hat{P}_{i+2}$ have the same x-coordinate.

(e) Basically, we need to show that the only way to get a cycle of size two is to satisfy the conditions $y_{i+1} > -y_{i+1}$ and $x_i \bmod s = x_{i+1} \bmod s$. It is not hard to see that the probability that $x_i \bmod s = x_{i+1} \bmod s$ is $1/s$ if the x-coordinates are random. Combined with the assumption that $y_{i+1} > -y_{i+1}$, the probability of meeting these conditions is $1/2s$.

8. (a) Since ϕ has degree q, according to Washington 3.15, the determinant of ϕ as an endomorhpism on $E[n]$ is $q \bmod n$. This determinant is the product of two eigenvalues α and β. Since $E(\mathbb{F}_q)$ has a point P of order n, $E[n]$ contains a point on $E(\mathbb{F}_q)$, which is fixed by ϕ. This means that P is an eigenvector of ϕ with eigenvalue 1. Thus, the other eigenvalue is q.

(b)

$$\hat{e}(P,P) = \hat{e}(\phi P, \phi P) = \hat{e}(P,P)^{\deg\phi} = \hat{e}(P,P)^q$$

Thus $\hat{e}(P,P) \in \mathbb{F}_q$. Since n is prime, if $\hat{e}(P,P)$ is not 1, it is a primitive nth root of 1, and thus all the nth roots of 1 are in $\mathbb{F}_q$, a contradiction. Therefore $\hat{e}(P,P) = 1$.

Similarly,

$$\hat{e}(Q,Q)^q = \hat{e}(\phi Q, \phi Q) = \hat{e}(qQ, qQ) = \hat{e}(Q,Q)^{q^2}$$

Thus, the order of $\hat{e}(Q,Q)^q$ divides $q-1$. By the same argument as above, this means $\hat{e}(Q,Q)^q = 1$. However, the order of $\hat{e}(Q,Q)$ is either n or 1, and n does not divide q, so $\hat{e}(Q,Q) = 1$

(c)

$$\begin{aligned} \hat{e}((1+\alpha)P, (1+\alpha)P) &= \hat{e}(P,P)\hat{e}(\alpha P, \alpha P)\hat{e}(P, \alpha P)\hat{e}(\alpha P, P) \\ &= \hat{e}(P,P)\hat{e}(P,P)^{\deg\alpha}\hat{e}(P,\alpha P)\hat{e}(\alpha P, P) \\ &= \hat{e}(P,\alpha P)\hat{e}(\alpha P, P) \end{aligned}$$

But

$$\hat{e}((1+\alpha)P, (1+\alpha)P) = \hat{e}(P,P)^{\deg(1+\alpha)} = 1$$

Thus $\hat{e}(P,\alpha P)) = \hat{e}(\alpha P, P)^{-1}$

(d) Since n is prime and P has order n, $\alpha(P)$ must have order n or order 1. However, since $\alpha(P) \notin \langle P \rangle$, $\alpha(P)$ must have order n. Thus, P and $\alpha(P)$ must span $E[n]$. Therefore, we can write $T = aP + b\alpha(P)$.

$$\hat{e}(T,T) = \hat{e}(aP + b\alpha(P), aP + b\alpha(P)) = \hat{e}(P,P)^{\deg(a+b\alpha)} = 1$$

(e)
$$1 = \hat{e}(S+T, S+T) = \hat{e}(S,S)\hat{e}(T,T)\hat{e}(S,T)\hat{e}(T,S) = \hat{e}(S,T)\hat{e}(T,S)$$

9. In lecture, we saw that if $E[r] \not\subseteq E(\mathbb{F}_q)$, then r divides $q^k - 1$ if and only if $E[r] \subset E(\mathbb{F}_{q^k})$.

(a)
$$q^3 - 1 = (q-1)(q^2+q+1) = (q-1)(q+\sqrt{q}+1)(q-\sqrt{q}+1)$$
By assumption, r divides $\#E(\mathbb{F}_q) = q + 1 \pm \sqrt{q}$, so r divides $q^3 - 1$

(b)
$$q^4 - 1 = (q^2-1)(q^2+1) = (q+1)(q-1)(q+\sqrt{2q}+1)(q-\sqrt{2q}+1)$$
By assumption, r divides $\#E(\mathbb{F}_q) = q + 1 \pm \sqrt{2q}$, so r divides $q^4 - 1$

(c)
$$q^6 - 1 = (q^3-1)(q^3+1) = (q^3-1)(q+1)(q+\sqrt{3q}+1)(q-\sqrt{3q}+1)$$
By assumption, r divides $\#E(\mathbb{F}_q) = q + 1 \pm \sqrt{3q}$, so r divides $q^6 - 1$